Study Guide AIHEC MATH Bowl

Note:

These are previous tests from AMATYC Student Mathematics League. Solutions to all these tests are included.

Although these are not the questions for the test, it should serve as a guideline to study for the AIHEC MATH Bowl.

As a reminder,

a. The first round will be an hour long and will consist of 10 problems to be answered individually by each member of the team. The team score will be calculated by dividing the sum of the team members' individual scores by 3 (even if the team has fewer than 3 members) and that will be the final score for that round.

b. The second and final round will consist of 10 problems to be answered as a team. In this round, interaction among team members is permitted and encouraged as they work together to solve the problems.

1. Hiromi buys a TV in Oregon (where there is no sales tax) and receives a 13% discount on the list price. Later she sees an ad offering a 19% discount. If the store agrees to refund the difference and Hiromi gets $21 back, what is the TV's list price?

A. $300 B. $320 C. $325 D. $330 E. $350

2. What is the coefficient of x^2 in the expansion of $(x^2 + 3x - 1)^2$?

A. -2 B. -1 C. 2 D. 7 E. 9

3. Find the sum of the values of x for which $\dfrac{x - 2}{x^2 - 4x + 3}$ is undefined.

A. 3 B. 4 C. 5 D. 6 E. 7

4. The lines with equations ax + by = c and dx + ey = f are perpendicular (a, b, c, d, e, f constants). Which of the following must be true?

A. ad - be = 0 B. ad + be = -1 C. ae + bd = -1 D. ae + bd = 0 E. ad + be = 0

5. A palindrome is a word or a number (like RADAR or 1221) which reads the same forwards and backwards. If dates are written in the format MMDDYY, how many dates in the 21st century are palindromes?

A. 1 B. 12 C. 24 D. 36 E. 144

6. In square ABCD, E is the midpoint of CD. Suppose AE intersects BD at F and the extension of side BC at G. If AF = 2005 and EF = 1000, find EG.

A. 1000 B. 2000 C. 2005 D. 3005 E. 4010

7. For positive values of x for which $\mathrm{Sec}^{-1}(x)$ is in the first quadrant, $\mathrm{Sec}^{-1}(x) =$

A. $\dfrac{1}{\mathrm{Cos}^{-1}(x)}$ B. $\sec\left(\dfrac{1}{x}\right)$ C. $\cos x$ D. $\cos\left(\dfrac{1}{x}\right)$ E. $\mathrm{Cos}^{-1}\left(\dfrac{1}{x}\right)$

8. Mrs. Abbott finds that the number of possible groups of 3 students in her class is exactly five times the number of possible groups of 2 students. How many students are in her class?

A. 15 B. 17 C. 20 D. 22 E. 25

9. In how many ways can slashes be placed among the letters AMATYCSML to separate them into four groups with each group including at least one letter?

A. 28 B. 56 C. 70 D. 84 E. 112

10. Two motorists set out at the same time to go from Danbury to Norwich, 100 miles apart. They follow the same route and travel at different but constant speeds of an integral number of miles per hour. The difference in their speeds is a prime number of miles per hour, and after driving for two hours, the distance of the slower car from Danbury is five times that of the faster car from Norwich. What is the faster car's speed?

A. 40 mph B. 42 mph C. 44 mph D. 46 mph E. 48 mph

11. The sum cos 1° + cos 2° + cos 3° + … + cos 357° + cos 358° + cos 359° is equal to

A. $\frac{□}{2}$ B. □ C. 0 D. 1 E. –1

12. If M = $\begin{bmatrix} 0 & 2 \\ 5 & 0 \end{bmatrix}$ and N = $\begin{bmatrix} 0 & 5 \\ 2 & 0 \end{bmatrix}$, find M^{2005}.

A. $10^{1002}M$ B. $10^{1002}N$ C. $10^{2004}M$ D. $10^{2004}N$ E. $10^{2005}M$

13. A basketball team scores 78 points on 41 baskets (field goals count 2 points, free throws 1 point, and 3-point shots 3 points). If the number of each type of basket is different, and the number of baskets of any two types differs by no more than 4, how many field goals are scored?

A. 11 B. 12 C. 13 D. 14 E. 15

14. Which of the following is a factor of $(10^{2005} + 1)^2 + (10^{2005} + 2)^2 - (10^{2005})^2$?

A. $10^{2005} - 1$ B. $10^{2005} + 3$ C. $10^{2005} + 4$ D. $10^{2005} + 5$ E. $10^{2005} + 6$

15. The volume of cylinder A is 108π, which is twice the volume of cylinder B. If the radius and height of A are the height and radius respectively of B, find the height of cylinder B.

A. 3 B. 4 C. 6 D. 9 E. 12

16. In how many ways can nine identical dominos (2x1 rectangles) be used to exactly cover a 3x6 rectangle with no overlap? Assume two coverings are different if the nine dominos are not in exactly the same positions.

A. 27 B. 31 C. 35 D. 41 E. 47

17. Two triangular regions are formed in the first quadrant, one with vertices (0,0), (5,0), and (0,12), the other with vertices (0,0), (8,0), and (0,6). Find the area to the nearest integer of the region they have in common.

A. 15 B. 17 C. 19 D. 21 E. 23

18. A triangle has sides of length a, b, and c, which are consecutive integers in increasing order, and $\cos C = \frac{5}{16}$. Find cos A.

A. $\frac{5}{8}$ B. $\frac{7}{11}$ C. $\frac{13}{20}$ D. $\frac{2}{3}$ E. $\frac{11}{16}$

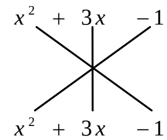
19. If $p > 5$ is a prime number, what is the largest integer which must be a factor of $p^4 - 1$?

A. 120 B. 150 C. 180 D. 240 E. 400

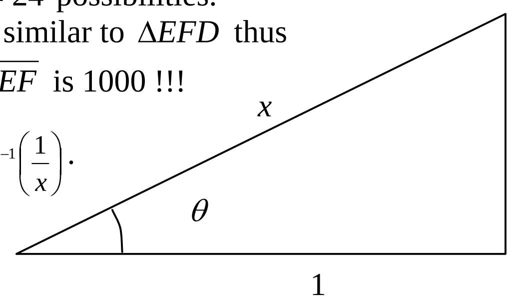
20. The circumradius of a triangle is the radius of the circle which contains all three of the triangle's vertices. The length of the circumradius of the triangle with sides of length 193, 194, and 195 is a rational number. Find this length to the nearest tenth.

A. 112.0 B. 112.1 C. 112.2 D. 112.3 E. 112.4

AMATYC February/March 2005



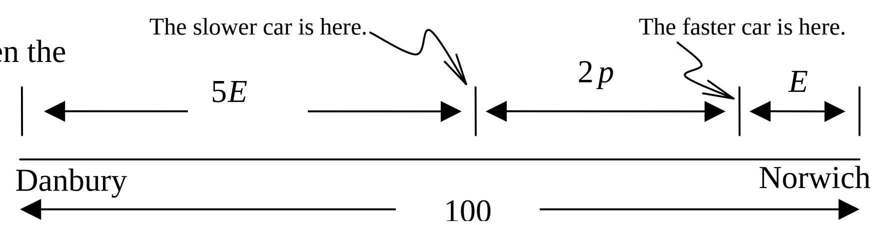
1. [E] $\$21 \div 6\% = \$21 \div 0.06 = \$350$
2. [D] $(1)(-1)+(3)(3)+(-1)(1) = 7$
3. [B] Want x such that $x^2 - 4x + 3 = 0$, which gives $x = 1, 3$. So the answer is $1+3=4$.
4. [E] The line $dx + ey = f$ is perpendicular to the line $ax + by = c$ if and only if $dx + ey = f$ is equivalent to $-bx + ay = k$ for some k. Namely, $dx + ey = f$ can be brought to the form $-bx + a = k$ when multiplied on both sides by a non-zero number. In particular, the ratio $(-b):a$ must equal the ratio $d:e$, thus $ad = (-b)e$, i.e. $ad + be = 0$. (Remark: Another way to think about this in the case where the line $ax + by = c$ is neither vertical nor horizontal is to use the condition $m_1 m_2 = -1$ for perpendicularity, and thus $\left(-\frac{a}{b}\right)\left(-\frac{d}{e}\right) = -1$, and so $ad + be = 0$.)
5. [C] MM can be any of 01, 02, 03, …, 10, 11, 12. After making the choice (e.g. 12), YY follows automatically (e.g. 21). So there are 12 possibilities for MM and YY together. DD can be either 11 or 22. So altogether there are $12 \times 2 = 24$ possibilities.
6. [NONE] This problem is erroneous. The triangle ΔAFB is similar to ΔEFD thus $\frac{\overline{AF}}{\overline{EF}} = \frac{\overline{AB}}{\overline{ED}} = 2$. Thus it is impossible for $\overline{AF} = 2005$ while $\overline{EF}$ is 1000 !!!



7. [E] Let $\theta = \sec^{-1} x$, then $\sec\theta = x$. So $\cos\theta = \frac{1}{x}$, i.e. $\theta = \cos^{-1}\left(\frac{1}{x}\right)$.

   We conclude that $\sec^{-1} x = \cos^{-1}\left(\frac{1}{x}\right)$

8. [B] $\binom{n}{3} = 5\binom{n}{2}$, i.e. $\frac{n(n-1)(n-2)}{1\times2\times3} = 5\cdot\frac{n(n-1)}{1\times2}$. Thus $\frac{n-2}{3} = 5$, so $n - 2 = 15$, $n = 17$.
9. [B] There are 9 slots in AMATYCSML. The problem can be interpreted as selecting 3 out of the 8 inter-slot spaces and placing a slash in each of the three. E.g. AMA/TY/C/SML, or A/MATYCS/M/L. Thus the answer is $\binom{8}{3} = \frac{8\times7\times6}{1\times2\times3} = 56$.
10. [B] Let the difference between the two speeds be p miles per hour, with p being a prime. After two hours, the situation is depicted in the accompanying picture.



Note that $5E$ is the number of miles traveled by the slower car in two hours, thus $5E$ is an even number. It follows that $E = 100 - 2p - 5E$ is an even number. From the picture, we see that $100 - 2p$ equals $6E$, and so is divisible by 12. That is, $2(50 - p)$ is divisible by 12, and so $50 - p$ is divisible by 6. It follows that p is even. But p is a prime, so $p = 2$. It then becomes a trivial matter to see that $6E = 96$, so $E = 16$, $5E = 80$, thus the speed of the slower car is $80 \text{ miles}/2\text{ hr} = 40\text{ mph}$, and hence the faster car has a speed of $40 + p = 40 + 2 = 42$ mph.

11. [E] Imagine 360 unit vectors emanating from the origin, pointing to the 360 points, including $(1, 0)$, evenly distributed on the unit circle centered at the origin. The sum of these 360 vectors is the null vector. Thus the sum of the x-components is 0, i.e. $\cos(0°)+\cos(1°)+\cos(2°)+\cdots+\cos(359°)=0$. It follows that $\cos(1°)+\cos(2°)+\cdots+\cos(359°)=-\cos(0°)=-1$

12. [A] Note that $\mathrm{M}^2=\mathrm{MM}=\begin{bmatrix}0 & 2\\ 5 & 0\end{bmatrix}\begin{bmatrix}0 & 2\\ 5 & 0\end{bmatrix}=\begin{bmatrix}10 & 0\\ 0 & 10\end{bmatrix}=10\mathrm{I}$, where $\mathrm{I}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$ is the identity matrix. So $\mathrm{M}^{2005}=\mathrm{M}^{2004}\,\mathrm{M}=(\mathrm{M}^2)^{1002}\,\mathrm{M}=(10\,\mathrm{I})^{1002}\,\mathrm{M}=10^{1002}\,\mathrm{M}$.

13. [C] Let there be x field goals, y free throws, and z 3-point shots. Then

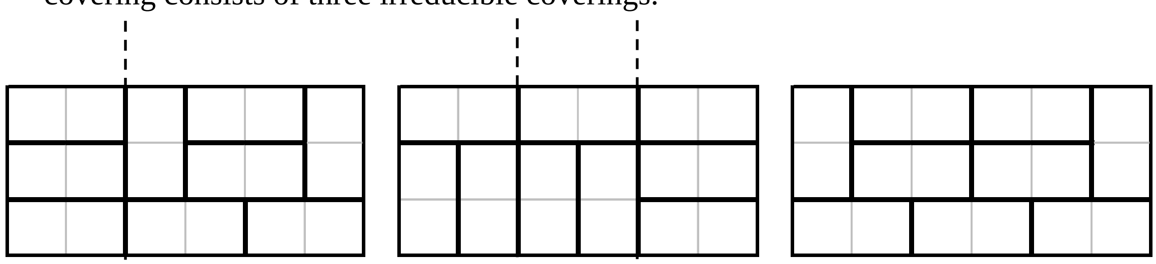
$$\begin{cases} x + y + z = 41 \\ 2x + y + 3z = 78 \end{cases}$$

Solve for y and z in terms of x to get $y=\dfrac{45-x}{2}$ and $z=\dfrac{37-x}{2}$. But the problem states that $|y-x|\le 4$, $|z-x|\le 4$, and $|y-z|\le 4$. These read, respectively, $\left|\dfrac{45-3x}{2}\right|\le 4$, $\left|\dfrac{37-3x}{2}\right|\le 4$, and $\left|\dfrac{8}{2}\right|\le 4$, where the last one is trivially true, while the first two can be written as $|45-3x|\le 8$ and $|37-3x|\le 8$. This means $37\le 3x\le 53$ and $29\le 3x\le 45$. It follows that $37\le 3x\le 45$. But $3x$ is divisible by 3. So $3x=39, 42, 45$, i.e. $x=13, 14, 15$. We rule out $x=14$ because it would make y, which is $\dfrac{45-x}{2}$, a non-integer. We further rule out $x=15$, because $y=\dfrac{45-x}{2}$ would be also 15, yet x, y, z are said to be different from each other. We are left with $x=13$ (with $y=16$, $z=12$).

14. [D] Let u stand for 10^{2005}, then the number at issue is $(u+1)^2+(u+2)^2-u^2=u^2+6u+5=(u+1)(u+5)$, Thus $u+5$, i.e. $10^{2005}+5$, is a factor.

15. [C] Suppose cylinder B has a height of h and a radius of r, then the problem says that $\pi r^2h=54\pi$ while $\pi h^2r=108\pi$. So $\dfrac{\pi r^2h}{\pi h^2r}=\dfrac{54\pi}{108\pi}$, i.e. $r=\dfrac{h}{2}$. Then $\pi h^2r=108\pi$ becomes $\dfrac{\pi h^3}{2}=108\pi$, so $h^3=216$, and so $h=6$.

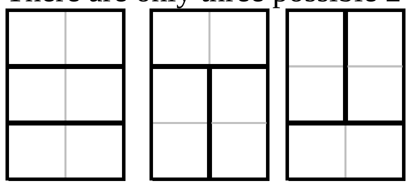
16. [D] Let's agree that we say a covering of rectangular matrix of squares by dominos is "irreducible" if it is impossible to make a vertical cut without breaking any domino. For example, the third covering below is irreducible, while the other two are reducible (meaning they are not irreducible). For a reducible case, once all possible vertical cuts of this sort are made, each of the resulting regions is an irreducible covering. E.g., the first covering below consists of two irreducible coverings, whereas the second covering consists of three irreducible coverings.



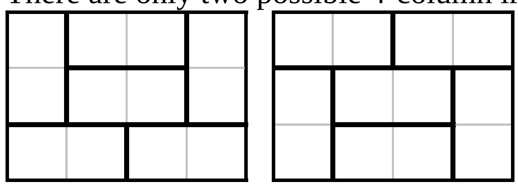
Convince yourself of the following assertions:

(1) An irreducible region can have only 2, 4, or 6 columns, because it is patched up by dominos and thus has to contain an even number of squares.

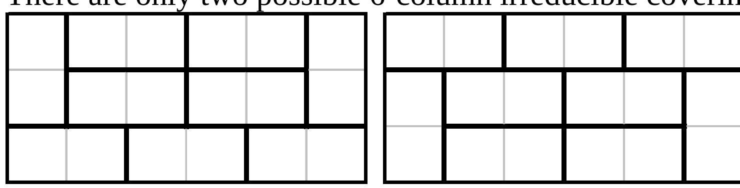
(2) There are only three possible 2-column irreducible coverings:



(3) There are only two possible 4-column irreducible coverings:



(4) There are only two possible 6-column irreducible coverings:



Now, we are set.

There are $3 \times 3 \times 3$ possible coverings of type "2-2-2" (A covering is of type "2-2-2" if it consists of three 2-column irreducible coverings.)

There are 2×3 possible coverings of type "4-2".

There are 3×2 possible coverings of type "2-4".

There are 2 possible coverings of type "6".

Altogether, there are therefore $27 + 6 + 6 + 2 = 41$ possible configurations.

17. [C] The line segment joining $(5, 0)$ and $(0, 12)$ falls on the line $\frac{x}{5} + \frac{y}{12} = 1$, i.e. $12x + 5y = 60$ The line segment joining $(8, 0)$ and $(0, 6)$ falls on the line $\frac{x}{8} + \frac{y}{6} = 1$, i.e. $3x + 4y = 24$. Solve the system

$\begin{cases} 12x + 5y = 60 \\ 3x + 4y = 24 \end{cases}$ to get $x = \frac{40}{11}$, $y = \frac{36}{11}$. The problem thus seeks to find the area of the quadrilateral with vertices $(0, 0)$, $(5, 0)$, $(\frac{40}{11}, \frac{36}{11})$, and $(0, 6)$, i.e. the sum of the areas of two triangles: one with vertices $(0, 0)$, $(5, 0)$, $(\frac{40}{11}, \frac{36}{11})$, the other with vertices $(0, 0)$, $(0, 6)$, $(\frac{40}{11}, \frac{36}{11})$. The former triangle has a base of 5 and a height of $\frac{36}{11}$. The latter triangle has a base of 6 and a height of $\frac{40}{11}$. So the answer is

$\frac{1}{2} \cdot 5 \cdot \frac{36}{11} + \frac{1}{2} \cdot 6 \cdot \frac{40}{11} = \frac{210}{11} = 19\frac{1}{11} \approx 19$.

18. [C] Use the law of cosines.

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab} = \frac{a^2 + (a+1)^2 - (a+2)^2}{2a(a+1)} = \frac{a^2 - 2a - 3}{2a(a+1)} = \frac{(a-3)(a+1)}{2a(a+1)} = \frac{a-3}{2a}.$$

But $\cos C = \frac{5}{16}$, so $\frac{a-3}{2a} = \frac{5}{16}$, and thus $a = 8$, so $b = 9$, $c = 10$. Then

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{9^2 + 10^2 - 8^2}{2(9)(10)} = \frac{13}{20}.$$

19. [D] $p^4 - 1 = (p^2 - 1)(p^2 + 1) = (p-1)(p+1)(p^2+1)$. Since $p > 5$ is a prime, we see p is an odd number. Thus $p-1$, $p+1$, and p^2+1 are all even numbers. Therefore $(p-1)(p+1)(p^2+1)$ is divisible by 8. But we can do better: $(p-1)(p+1)(p^2+1)$ is divisible by 16. We will prove this by showing that among the even numbers $p-1$, $p+1$, and p^2+1, at least one is actually a multiple of 4. First note that, since p is odd, we have either $p \equiv 1 \bmod 4$ or $p \equiv 3 \bmod 4$

- If $p \equiv 1 \bmod 4$, then $p-1$ is divisible by 4.
- If $p \equiv 3 \bmod 4$, then $p+1$ is divisible by 4.

In either case $(p-1)(p+1)(p^2+1)$ is divisible by 16.

Now consider mod 3: we have either $p \equiv 1 \bmod 3$ or $p \equiv 2 \bmod 3$. (Since p is a prime greater than 5, it is impossible to have $p \equiv 0 \bmod 3$.).

- If $p \equiv 1 \bmod 3$, then $p-1$ is divisible by 3.
- If $p \equiv 2 \bmod 3$, then $p+1$ is divisible by 3.

Thus $(p-1)(p+1)(p^2+1)$ is divisible by 3.

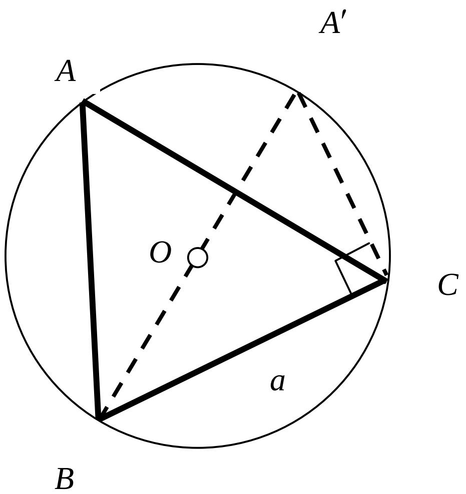
Likewise, consider mod 5: we have $p \equiv -2 \bmod 5$, $p \equiv -1 \bmod 5$, $p \equiv 1 \bmod 5$, or $p \equiv 2 \bmod 5$. (Since p is a prime greater than 5, it is impossible to have $p \equiv 0 \bmod 5$.).

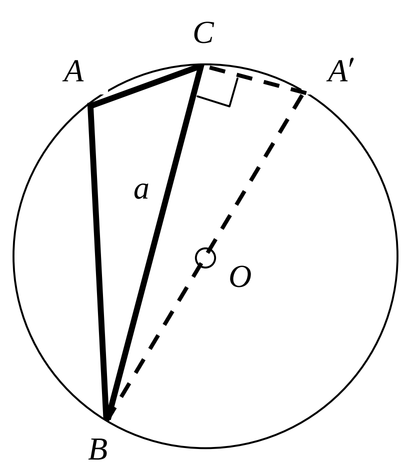
- If either $p \equiv -2 \bmod 5$ or $p \equiv 2 \bmod 5$, then $p^2 + 1 \equiv 4 + 1 \bmod 5$, i.e. p^2+1 is divisible by 5.
- If $p \equiv -1 \bmod 5$, then $p+1$ is divisible by 5.
- If $p \equiv 1 \bmod 5$, then $p-1$ is divisible by 5.

Thus $(p-1)(p+1)(p^2+1)$ is divisible by 5.

Now that $(p-1)(p+1)(p^2+1)$ is divisible by 16, 3, and 5, it follows that 240 is a factor of $(p-1)(p+1)(p^2+1)$ for all prime p greater than 5. To see that no integer greater than 240 can achieve this, simply consider two special cases: When $p = 7$, $(p-1)(p+1)(p^2+1) = 6 \times 8 \times 50 = 240 \times 10$, whereas when $p = 11$, $(p-1)(p+1)(p^2+1) = 10 \times 12 \times 122 = 240 \times 61$.

20. [A] First of all, we have to know the fact that the common ratio $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$ in the well-known law of sines is actually $2R$, where R is the circumradius.





To see this, consider the case shown in the first picture. Recall from geometry that the two inscribed angles $\angle BAC$ and $\angle BA'C$ are congruent, so $\dfrac{a}{\sin(\angle BAC)} = \dfrac{a}{\sin(\angle BA'C)}$.

But $\overline{A'B}$ is a diameter, therefore $\angle BCA'$ is a right angle. So $\dfrac{a}{\sin(\angle BA'C)} = \overline{A'B} = 2R$.

Therefore $\dfrac{a}{\sin(\angle BAC)} = 2R$. For the scenario shown in the second picture, the measures of the inscribed angles $\angle BAC$ and $\angle BA'C$ sum to $180°$, because the measure of $\angle BA'C$ equals half of the measure of $\angle BOC$, whereas the measure of $\angle BAC$ equals half of the measure of the reflex angle (i.e. between $180°$ and $360°$) formed by $\angle BOC$. Thus $\sin(\angle BAC) = \sin(\angle BA'C)$. Again, $\dfrac{a}{\sin(\angle BA'C)} = \overline{A'B} = 2R$, so

$$\frac{a}{\sin(\angle BAC)} = 2R.$$

Secondly, we have to recall a formula for the area of the triangle ΔABC:

Area $= \frac{1}{2}bc\sin A$. We thus have $R = \dfrac{1}{2}(2R) = \dfrac{1}{2}\cdot\dfrac{a}{\sin A} = \dfrac{abc}{2bc\sin A} = \dfrac{abc}{4\cdot \text{Area}}$. Finally, we have to invoke Heron's formula for the area of a triangle:

Area $= \sqrt{s(s-a)(s-b)(s-c)}$, where $s = \dfrac{a+b+c}{2} = \dfrac{193+194+195}{2} = 291$. Thus

$$R = \frac{abc}{4\cdot \text{Area}} = \frac{abc}{4\cdot\sqrt{s(s-a)(s-b)(s-c)}} = \frac{abc}{4\cdot\frac{1}{4}\sqrt{(a+b+c)(b+c-a)(a+c-b)(a+b-c)}}$$

$$= \frac{(193)(194)(195)}{\sqrt{(582)(196)(194)(192)}} = \frac{(193)(194)(195)}{\sqrt{(3)(194)(196)(194)(192)}} = \frac{(193)(195)}{\sqrt{(3)(196)(192)}} = \frac{(193)(195)}{\sqrt{(3)(4)(49)(3)(64)}}$$

$$= \frac{(193)(195)}{(3)(2)(7)(8)} = \frac{(193)(65)}{(2)(7)(8)} = \frac{12545}{112} = 112\tfrac{1}{112} \approx 112.0$$

1. If $f(x) = \cos \pi x$ and $g(x) = 2x$, find $f(g(1)) - g(f(1))$.

A. -3 B. -1 C. 0 D. 1 E. 3

2. How many different four-digit numbers can be formed by arranging the digits 2, 0, 0, and 6?

A. 6 B. 8 C. 10 D. 12 E. 24

3. If ABCD, DCEF, and FEGH are squares with A, B, C, D, E, F, G, and H all distinct points, find $m\angle GAH + m\angle GDH + m\angle GFH$ to the nearest tenth of a degree.

A. 80° B. 87.5° C. 90° D. 92.5° E. 100°

4. I sold a horse for \$200, losing 20%. I bought another horse and sold it for a 25% profit. If I broke even on the two transactions together, what was the total cost of the two horses?

A. \$432 B. \$450 C. \$500 D. \$540 E. \$562.50

5. Let A(m,n) be the set of n consecutive positive integers whose least element is m. What is the greatest integer in $A(17,49) \cap A(49,17)$?

A. 33 B. 49 C. 65 D. 66 E. 67

6. Let $a, b > 0$, $M = \sum_{n=1}^{k} \ln(an) - \sum_{n=1}^{k} \ln(bn)$, $N = e^M$, and $P = \sqrt[k]{N}$. Then P equals

A. $\frac{a}{b}$ B. $a - b$ C. $\sqrt[k]{k(a-b)}$ D. $\sqrt[k]{\frac{ka}{b}}$ E. $e^{a/bk}$

7. Which of the following imply that the real number x must be rational?

I. x^5, x^7 are both rational
II. x^6, x^8 are both rational
III. x^5, x^8 are both rational

A. I, II only B. I, III only C. II, III only D. III only
E. none of these combinations

8. A positive integer less than 1000 is chosen at random. What is the probability it is a multiple of 3, but a multiple of neither 2 nor 9 ?

A. $\frac{1}{10}$ B. $\frac{1}{9}$ C. $\frac{1}{8}$ D. $\frac{2}{9}$ E. $\frac{1}{3}$

9. Let r and s be the solutions to the equation $x^2 + 3x + c = 0$. If $r^2 + s^2 = 33$, find the value of c.

A. -21 B. -12 C. 1 D. 12 E. 21

10. Joe must determine the greatest whole number of feet from which a ceramic ball can be dropped without breaking. He has two identical ceramic balls which he can drop from any whole numbered height he wants. If he must determine this height with no more than 12 drops, what is the greatest height for which he can determine this with certainty?

A. 20-25 ft B. 26-40 ft C. 41-50 ft D. 51-75 ft E. more than 75 ft

11. In convex pentagon AMTYC, $\overline{CY} \perp \overline{YT}$, $\overline{MT} \perp \overline{YT}$, CY = YT = 63, MT = 79, AM = 39, and AC = 52. Find the area of the pentagon.

A. 5487 B. 5500 C. 5525 D. 5600 E. 5624

12. The *midrange* of a set of numbers is the average of the greatest and least values in the set. For a set of six increasing nonnegative integers, the mean, the median, and the midrange are all 5. How many such sets are there?

A. 10 B. 12 C. 20 D. 24 E. 30

13. The sum of the absolute values of all solutions of the equation $\left|x^3 + 4x^2 - 6x - 22\right| = x^2 + 2x + 2$ can be written in the form $a + b\sqrt{c}$, c a prime. Find a + b + c.

A. 12 B. 14 C. 16 D. 17 E. 18

14. Find the number of three-digit numbers containing no even digits which are divisible by 9.

A. 8 B. 9 C. 10 D. 11 E. 12

15. If θ is the acute angle formed by the lines with equations y = 2x - 5 and y = 1 - 3x, find tan θ.

A. $\frac{1}{\sqrt{3}}$ B. $\frac{1}{2}$ C. 1 D. 2 E. $\sqrt{3}$

16. Find the number of points of intersection of the unit circle and the graph of the equation $y^2 - xy - |x|y + x|x| = 0$.

A. 0 B. 1 C. 2 D. 3 E. 4

17. Suppose that for a function y = f(x), f(x) > x for all x. Let A be the point with x-coordinate a on the function y = f(x) and B be the point on the graph of the line y = x for which $\overline{AB}$ is perpendicular to the line. Find an expression for the distance from A to B.

A. $(f(a) - a)\sqrt{2}$ B. $a\frac{\sqrt{2}}{2}$ C. $(f(a) - a)\frac{\sqrt{2}}{2}$ D. $f(a) - a$ E. $f(a)\sqrt{2}$

18. In the quadrilateral PQRS, PQ = 1, QR = RS = $\sqrt{2}$, PS = $\sqrt{3}$, and QS = 2. If T is the point of intersection of the diagonals, find the measure in degrees of angle RTS.

A. 45 B. 55 C. 60 D. 75 E. 105

19. Call a composite number *circumfactorable* if all of its positive integer factors greater than 1 can be placed around a circle so that any two adjacent factors have a common factor greater than 1. How many composite numbers less than 200 are not circumfactorable?

A. 50 B. 52 C. 54 D. 56 E. 58

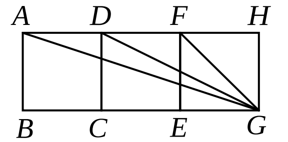
20. A circle contains 2006 points chosen so that the arcs between any two adjacent points are equal. Three of these points are chosen at random. Let the probability that the triangle formed is right be R, and the probability that the triangle formed is isosceles be I. Find $|R - I|$.

A. 0 B. 1/5 C. 1/4 D. 1/3 E. 1/2

AMATYC Contest (Spring 2006) SOLUTIONS

1. [E] $f(g(1))-g(f(1))=f(2)-g(-1)=1-(-2)=3$.

2. [A] The thousands place has to be 2 or 6, then exactly one of the remaining places is the other nonzero digit. The two places left are 0s. This gives $(2)(3)=6$ possibilities.



3. [C] $\tan(m\angle GAH+m\angle GDH)=\dfrac{\tan(m\angle GAH)+\tan(m\angle GDH)}{1-\tan(m\angle GAH)\cdot\tan(m\angle GDH)}=\dfrac{\frac{1}{3}+\frac{1}{2}}{1-\frac{1}{3}\cdot\frac{1}{2}}=1$.

   Therefore $m\angle GAH+m\angle GDH=45°$. The answer follows.

4. [B] The cost of the first horse is $\$200/80\%=\$250$, and the loss of selling is \$50. Thus the profit from selling the second horse is also \$50, therefore the cost of the second horse is $\$50/25\%=\$200$. The answer is $\$250+\$200=\$450$.

5. [C] $A(17,49)=\{17,18,\cdots,65\}$, $A(49,17)=\{49,50,\cdots,65\}$. The answer follows.

6. [A] $M=\sum_{n=1}^{k}[(\ln a+\ln n)-(\ln b+\ln n)]=\sum_{n=1}^{k}(\ln a-\ln b)=k(\ln a-\ln b)=k\ln\frac{a}{b}$.

   $N=e^{k\ln(a/b)}=(e^{\ln(a/b)})^k=(\frac{a}{b})^k$. The answer follows.

7. [B] The key is that 5 and 7 are coprime, so are 5 and 8, whereas 6, 8 are not coprime. So, if I is true, then $(x^5)^3(x^7)^{-2}=x$ is rational. Likewise, if III is true, then $(x^5)^5(x^8)^{-3}=x$ is rational. Whereas, with $x=\sqrt{2}$, II would be true while x is not rational.

8. [B] Let A_k be the set of all positive integers less than 1000 that are divisible by k. Let $|S|$ stand for the number of elements in set S. We need $|A_3|-|A_3\cap A_2|-|A_3\cap A_9|+|A_3\cap A_2\cap A_9|$ $=|A_3|-|A_6|-|A_9|+|A_{18}|=333-166-111+55=111$. So the answer is $\frac{111}{999}=\frac{1}{9}$.

9. [B] Since r and s are the two solutions to $x^2+3x+c=0$, we have $r+s=-3$ and $rs=c$. Now, $2rs=(r+s)^2-(r^2+s^2)=(-3)^2-33=-24$. So $c=rs=-12$.

10. [E] Drop at 12 ft. If it breaks, then, with 11 trials remaining, drop at 1ft, then 2 ft, then 3ft, etc. until it breaks or until the trials run out. Else (if dropping at 12 ft didn't break the ball) jump to drop at $12+11=23$ ft. If it breaks, then, with 10 trails remaining, drop at $12+1=13$ ft, $12+2=14$ ft, $12+3=15$ ft, etc. until it breaks or until the trials run out. Else (if dropping at $12+11=23$ ft didn't break the ball) jump to drop at $12+11+10=33$ ft. If it breaks, then, with 9 trials remaining, drop at $12+11+1=24$ ft, $12+11+2=25$ ft, $12+11+3=26$ ft, etc. until it breaks or until the trials run out. Continue this way. Such a strategy can determine with certainty the greatest whole number of feet from which a ball can be dropped without breaking provided it is no greater than $12+11+10+9+\cdots+3+2=(12+2)(11)/2=77$.

11. [A] Let B be the point on $\overline{MT}$ such that $\overline{CB}\perp\overline{MT}$. Then $CB=63$, $MB=16$. The Pythagorean Theorem gives $CM=\sqrt{63^2+16^2}=65$. Then ΔAMC is a 5:4:3 right triangle (as 65:52:39 is 5:4:3.) The area of the pentagon is the area of the trapezoid $CMTY$ plus the area of the right triangle ΔAMC. This gives $\frac{1}{2}(63+79)(63)+\frac{1}{2}(52)(39)=5487$.

12. [A] Let the increasing nonnegative integers be x_1, x_2, x_3, x_4, x_5, x_6. Now, 5 is halfway between x_3 and x_4 as the median is 5. Likewise, 5 is halfway between x_1 and x_6 as the midrange is 5. The mean is also 5, so 5 has to be halfway between x_2 and x_5. Thus we only have to enumerate the possible values of x_1, x_2, x_3. They are from 0, 1, 2, 3, 4, and so the answer is $C_3^5=10$.

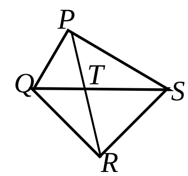
13. [E] First, note that x^2+2x+2 is $(x+1)^2+1$ and so it is always positive. For the case $x^3+4x^2-6x-22=x^2+2x+2$, we have $x^3+3x^2-8x-24=0$, i.e. $(x+3)(x^2-8)=0$. For the case $x^3+4x^2-6x-22=-(x^2+2x+2)$, we have $x^3+5x^2-4x-20=0$, i.e. $(x+5)(x^2-4)=0$. So the solutions are -3, $2\sqrt{2}$, $-2\sqrt{2}$, -5, 2, -2. The sum of their absolute values is $3+2\sqrt{2}+2\sqrt{2}+5+2+2=12+4\sqrt{2}$. The answer follows.

14. [D] The sum of the three digits are from 1, 3, 5, 7, 9. Their sum has to be divisible by 9, and so can be 9, 18, 27. For a sum of 9, it has to be 711, 531, 333, or numbers resulting from rearranging the digits – there are $3+3!+1=10$ possibilities. A sum of 18 is impossible, as all three digits are odd. A sum of 27 comes from only 999. So we have a total of 11 possibilities.

15. [C] $\alpha=\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{3}$. So $\tan\alpha=\dfrac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{2}\cdot\frac{1}{3}}=1$.

16. [D] If $x\ge 0$, then $y^2-2xy+x^2=0$, i.e. $(y-x)^2=0$, so $y=x$, thus $(x,y)=(1/\sqrt{2},1/\sqrt{2})$. If $x<0$, then $y^2-x^2=0$, so $y=\pm x$, and so $(x,y)=(-1/\sqrt{2},-1/\sqrt{2})$ or $(-1/\sqrt{2},1/\sqrt{2})$.

17. [C] $A=(a,f(a))$. Let C be the mirror image of A with respect to the line $y=x$. Thus $C=(f(a),a)$. Then B is the midpoint of $\overline{AC}$. The vertical line through A and the horizontal line through C meet at a point D on the line $y=x$, with ΔADC being an isosceles right triangle. $AD=f(a)-a$, so $AC=(f(a)-a)\sqrt{2}$ Thus $AB=(f(a)-a)\dfrac{\sqrt{2}}{2}$.



18. [D] ΔSQR is an isosceles right triangle, ΔSQP is a $30°$-$60°$-$90°$ right triangle. Thus the points P, Q, R, S fall on a circle with $\overline{SQ}$ being a diameter. Therefore $\angle RTS=\angle TPS+\angle TSP=\angle RPS+\angle TSP=\angle RQS+\angle TSP=45°+30°=75°$.

19. [D] A composite number is not circumfactorable precisely when it is of the form p_1p_2, where p_1 and p_2 are distinct primes. To prove this, observe that such a p_1p_2 is indeed not circumfactorable.. Also observe that a composite number of the form ${p_1}^2$, ${p_1}^2p_2$ or $p_1p_2p_3$ is circumfactorable, where p_1, p_2, p_3 are distinct primes. To complete the proof, we only have to show that if m is a circumfactorable composite number then, for a prime p, the number mp is also circumfactorable. To do this, let all the factors of m that are greater than 1 be arranged around a circle so that any two adjacent factors have a common factor greater than 1. Call such an arrangement “permissible.” Suppose p itself is a factor of m. A factor greater than 1 of mp must fall into one of the following two mutually exclusive cases: (1) It is already a factor of m, and so already on the circle; (2) It is not a factor of m but is of the form ap, where $a>1$ is a factor of m already on the circle; For each factor ap of mp that falls in case (2), place ap right next to a. (Either side is okay.) This results in a permissible arrangement for mp. If p itself is not a factor of m, then in addition to (1) and (2) there will also be: case (3) p. In this situation, for each factor of m originally on the circle there is exactly one factor of mp that falls into case (2). Though we can place ap on either side of a, let’s assume that we make sure at least two factors a_1, a_2 of m originally adjacent on the circle have their case (2) counterparts a_1p, a_2p placed adjacent to each other. With this, we deal with the remaining factor p at the very end of the process by placing it right between a_1p and a_2p. This concludes the proof. We now count composite numbers, less than 200, of the form p_1p_2, where $p_1<p_2$ are distinct primes. Note that p_1 can only be 2, 3, 5, 7, 11, 13. For each p_1, the prime p_2 is greater than p_1 and can run up to a certain value. For example for $p_1=2$, we have p_2 being any of the 24 primes 3, 5, 7, 11, …, 97 (Of course, we have to memorize all prime numbers less than 100.) In the end, we enumerate all possibilities, and it works out to be $24+16+9+5+2+0=56$

20. [A] There are $(1003)(2004)$ possible right triangles. This is because the hypotenuse must be a diameter, which has 1003 possibilities, for each of which there are 2004 choices for the remaining vertex. There are $(2006)(1002)$ possible isosceles triangles. To see this, first observe that such an isosceles triangle cannot be equilateral because 2006 is not divisible by 3. Thus it has a distinguished vertex – the one where the two sides of equal length meet. This vertex can be any of the 2006 points. For each choice, there will be 1002 possible way to choose the opposite side. Since $(1003)(2004)=(2006)(1002)$, it follows that $I=R$.

1. Line L has equation $y = 2x + 3$, and line M has the same y-intercept as L. Which of the points below must M contain to be perpendicular to L?

A. (-4, 5) B. (-2, 5) C. (-1, 5) D. (1, 5) E. (4, 5)

2. Sue just received a 5% raise. Now she earns \$1200 more than Lisa. Before Sue's raise, Lisa's salary was 1% higher than Sue's. What is Lisa's salary?

A. \$28,000 B. \$29,400 C. \$30,000 D. \$30,300 E. \$31,200

3. If $x = -1$ is one solution of $ax^2 + bx + c = 0$, what is the other solution?

A. $x = -a/b$ B. $x = -b/a$ C. $x = b/a$ D. $x = -c/a$ E. $x = c/b$

4. Ryan told Sam that he had 9 coins worth 45¢. Sam said, "There is more than one possibility. How many are pennies?" After Ryan answered truthfully, Sam said, "Now I know what coins you have." How many nickels did Ryan have?

A. 0 B. 3 C. 4 D. 5 E. 9

5. A point (a, b) is a lattice point if both a and b are integers. It is called *visible* if the line segment from (0, 0) to it does NOT pass through any other lattice points. Which of the following lattice points is visible?

A. (28, 14) B. (28, 15) C. (28, 16) D. (28, 18) E. (28, 21)

6. A flea jumps clockwise around a clock starting at 12. The flea first jumps one number to 1, then two numbers to 3, then three to 6, then two to 8, then one to 9, then two, then three, etc. What number does the flea land on at his 2008th jump?

A. 4 B. 5 C. 6 D. 7 E. 8

7. In quadrilateral $ABCD$, E is the midpoint of $\overline{AB}$, F is the midpoint of $\overline{BC}$, G is the midpoint of $\overline{CD}$, and H is the midpoint of $\overline{DA}$. Which of the following must be true?

A. $\angle FEH = \angle FGH$ B. $\angle FEH = \angle EHG$ C. $\angle FEH + \angle EHG = 180°$
D. both A and C E. both B and C

8. All nonempty subsets of {2, 4, 5, 7} are selected. How many different sums do the elements of each of these subsets add up to?

A. 10 B. 11 C. 12 D. 14 E. 15

9. Luis solves the equation $ax - b = c$, and Anh solves $bx - c = a$. If they get the same correct answer for x, and a, b, and c are distinct and nonzero, what must be true?

A. $a + b + c = 0$ B. $a + b + c = 1$ C. $a + b = c$ D. $b = a + c$ E. $a = b + c$

10. How many asymptotes does the function $f(x) = \dfrac{x^2 - 22x + 40}{x^2 + 13x - 30}$ have?

A. 0 B. 1 C. 2 D. 3 E. 4

11. Replace each letter of AMATYC with a digit 0 through 9 (equal letters replaced by equal digits, different letters replaced by different digits). If the resulting number is the largest such number divisible by 55, find A + M + A + T + Y + C.

A. 36 B. 38 C. 40 D. 42 E. 44

12. The equation $a^6 + b^2 + c^2 = 2009$ has a solution in positive integers a, b, and c in which exactly two of a, b, and c are powers of 2. Find $a + b + c$.

A. 43 B. 45 C. 47 D. 49 E. 51

13. ACME Widget employees are paid every other Friday (i. e., on Fridays in alternate weeks). The year 2008 was unusual in that ACME had 3 paydays in February. What is the units digit of the next year in which ACME has 3 February paydays?

A. 0 B. 2 C. 4 D. 6 E. 8

14. Five murder suspects, including the murderer, are being interrogated by the police. Results of a polygraph indicate two of them are lying and three are telling the truth. If the polygraph results are correct, who is the murder?

Suspect A: "D is the murderer" Suspect B: "I am innocent" Suspect C: "It wasn't E"
Suspect D: "A is lying" Suspect E: "B is telling the truth"

A. A B. B C. C D. D E. E

15. Two arithmetic sequences are multiplied together to produce the sequence 468, 462, 384, ... What is the next term of this sequence?

A. 250 B. 286 C. 300 D. 324 E. 336

16. In ΔABC, $AB = 5$, $BC = 9$, and $AC = 7$. Find the value of $\dfrac{\tan\frac{A-B}{2}}{\tan\frac{A+B}{2}}$.

A. $\frac{1}{8}$ B. $\frac{7}{9}$ C. $\frac{3}{2}$ D. $\frac{9}{7}$ E. 8

17. A pyramid has a square base 6 m on a side and a height of 9 m. Find the volume of the portion of the pyramid which lies above the base and below a plane parallel to the base and 3 m above the base.

A. 32 m^3 B. 36 m^3 C. 64 m^3 D. 72 m^3 E. 76 m^3

18. In ΔABC, $AB = AC$ and in ΔDEF, $DE = DF$. If AB is twice DE and $\angle D$ is twice $\angle A$, then the ratio of the area of ΔABC to the area of ΔDEF is:

A. $\tan A$ B. $2 \sec A$ C. $\csc 2A$ D. $\sec A \tan A$ E. $\cot 2A$

19. In hexagon $PQRSTU$, all interior angles = 120°. If $PQ = RS = TU = 50$, and $QR = ST = UP = 100$, find the area of the triangle bounded by QT, RU, and PS to the nearest tenth.

A. 1082.5 B. 1082.9 C. 1083.3 D. 1083.5 E. 1083.9

20. For all integers $k \ge 0$, $P(k) = \left(2^2+2^1+1\right)\left(2^2-2^1+1\right)\left(2^4-2^2+1\right)\cdots\left(2^{2^{k+1}}-2^{2^k}+1\right) - 1$ is always the product of two integers n and $n + 1$. Find the smallest value of k for which $n + (n + 1) \ge 10^{1000}$.

A. 9 B. 10 C. 11 D. 12 E. 13

AMATYC Contest (Fall 2008) SOLUTIONS

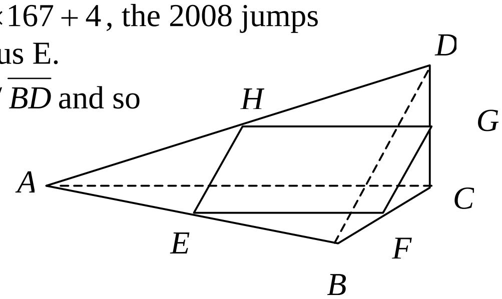
1. [A] Line M has equation $y = -\frac{1}{2}x + 3$. The answer follows.
2. [D] Lisa's salary is 101% of Sue's original salary. Sue's new salary is 105% of Sue's original salary. The difference, \$1200, is thus 4% of Sue's original salary, which is then $\$1200/0.04 = \$30{,}000$. Lisa's salary is hence $(\$30{,}000)(1.01) = \$30{,}300$.
3. [D] Note that the two roots (real or not, and with a double root counted as two) of a quadratic equation $x^2 + px + q = 0$ have a sum of $-p$ and a product of q. The quadratic equation at hand is $x^2 + \frac{b}{a}x + \frac{c}{a} = 0$, so the answer follows.
4. [E] Let there be x quarters, y dimes, z nickels, and w pennies. Thus

$$\begin{cases} 25x + 10y + 5z + w = 45 \\ x + y + z + w = 9 \end{cases}$$

From the first, we see w is divisible by 5. With $w = 5u$, rewrite the system as

$$\begin{cases} 5x + 2y + z + u = 9 \\ x + y + z + 5u = 9 \end{cases}$$

the second equation of which implies $u = 0, 1$. If $u = 0$, then $5x + 2y + z = 9$, and $x + y + z = 9$. The difference of the two gives $4x + y = 0$, and so x and y can only be 0, leaving $z = 9$. You may stop here and choose E. For completeness, let's check $u = 1$, which would give $5x + 2y + z = 8$, $x + y + z = 4$, thus $4x + y = 4$, leaving two possibilities: $x = 1$, $y = 0$, and thus $z = 3$, or $x = 0$, $y = 4$, $z = 0$. Thus Sam wouldn't have been able to know what coins Ryan has.
5. [B] Phrased differently, the two numbers are relatively prime. The answer follows.
6. [E] The jumps are 1, 2, 3, 2, 1, 2, 3, 2, …, with the sequence "1, 2, 3, 2" repeated indefinitely, each time giving a net move of $240°$ clockwise, namely $120°$ counterclockwise. Three of such 4-jump sequences (12 steps altogether) bring the flea back to the original position. As $2008 = 12 \times 167 + 4$, the 2008 jumps have the same effect as the last four, "1, 2, 3, 2". Thus E.
7. [D] Consider ΔABD, we see $\overline{EH} // \overline{BD}$. Likewise, $\overline{FG} // \overline{BD}$ and so $\overline{EH} // \overline{FG}$. Similarly, $\overline{EF} // \overline{HG}$. So $EFGH$ forms a parallelogram. The answer follows.



8. [C] Enumerating all nonempty subsets and calculating the corresponding sums would do.
9. [A] It means $\frac{b+c}{a} = \frac{a+c}{b}$, and so $a(a+c) = b(b+c)$, i.e. $a^2 - b^2 + ac - bc = 0$, so $(a-b)(a+b) + (a-b)c = 0$, so $(a-b)(a+b+c) = 0$. Since $a \neq b$, it follows that $a + b + c = 0$.
10. [C] $f(x) = \frac{(x-2)(x-20)}{(x-2)(x+15)} = \frac{x-20}{x+15}$. So $x = -15$ and $y = 1$ are the only asymptotes.
11. [C] To make AMATYC as large as possible, let's look for those of the form 989TYC. Since $989{,}999 = 55 \times 17999 + 54$, we see 989,945 is divisible by 55, but it's not of the form AMATYC. Repeatedly subtracting 55 to get 989890, 989835, 989780, and finally 989725, which is it. The answer follows: $9 + 8 + 9 + 7 + 2 + 5 = 40$

12. [E] Since $4^6 = 4096 > 2009$, a can only be 1, 2 or 3.

(a) $a = 3$, $b^2 + c^2 = 2009 - 3^6 = 1280$. By repeated long divisions by 2, write 1280 in binary form as $1{,}01{,}00{,}00{,}00{,}00_2$. So $1280 = 2^{10} + 2^8 = 32^2 + 16^2$, with $a + b + c = 3 + 32 + 16 = 51$. You may choose E at this point. But let's also double-check the other two cases.

(b) $a = 2$, $b^2 + c^2 = 2009 - 2^6 = 1945$. Since $a = 2$ is a power of 2, exactly one of b, c would be a power of 2. Say b. So $b = 1$, 2, 4, 8, 16, 32, none of which would make $1945 - b^2$ a perfect square.

(c) $a = 1$, $b^2 + c^2 = 2009 - 1^6 = 2008$. Since $a = 1$ is a power of 2, a similar reasoning would rule out this scenario. Even if you don't consider $a = 1$ as a power of 2, you can still rule out this case as follows: If $2008 = b^2 + c^2$ with b and c being powers of 2, then 2008 in binary form would have only one or two digits as 1, yet $2008 = 1{,}11{,}11{,}01{,}10{,}00_2$.

13. [C] We need a February with 5 Fridays. From the 1st Friday to the 5th inclusive, there would be 29 days, necessitating a leap year with February 1 falling on a Friday. From the year y to the year $y + 1$, the day of the week February 1 falls on shifts forward by 1 (because $365 = 7 \times 52 + 1$) except when the year y is a leap year, in which case it shifts by 2. Thus every four years it shifts by 5. So February 1 falls on a Friday again after 7 four-year periods, i.e. the year $2008 + 7 \times 4 = 2036$. However, the year 2036 would not work, because Friday, February 1, 2036 is $52 \times 28 + 7 \times 5 \div 7 = 1461$ weeks following a payday (Friday, February 1, 2008), therefore is not a payday. We thus have to wait until $2036 + 7 \times 4 = 2064$.

14. [E] Call the four statements A, B, C, D, E. Since suspect E sides with suspect B, so statements B, E are both true or both false. Statements A and D negate each other, so exactly one of them is false. With a total of 2 false statements, it follows that both statements E and B are true. To make a total of 2 false statements, C has to be false. Done.

15. [(Freebie!!)] The two arithmetic sequences are a, $a + x$, $a + 2x$, … and b, $b + y$, $b + 2y$, …. When multiplied together, we get the sequence

$ab = 468$, $ab + (bx + ay) + xy = 462$, $ab + 2(bx + ay) + 4xy = 384$, …

Thus $(bx + ay) + xy = 462 - 468 = -6$, and $(bx + ay) + 3xy = 384 - 462 = -78$.

Solve this linear system to get $bx + ay = 30$, and $xy = -36$. So the next term is

$(a + 3x)(b + 3y) = ab + 3(bx + ay) + 9xy = 468 + 3(30) + 9(-36) = 234$

16. [A] $\dfrac{\tan\frac{A-B}{2}}{\tan\frac{A+B}{2}} = \dfrac{\cos\frac{A+B}{2}\sin\frac{A-B}{2}}{\sin\frac{A+B}{2}\cos\frac{A-B}{2}} = \dfrac{\frac{1}{2}(\sin A - \sin B)}{\frac{1}{2}(\sin A + \sin B)} = \dfrac{\sin A - \sin B}{\sin A + \sin B} = \dfrac{a - b}{a + b} = \dfrac{9 - 7}{9 + 7} = \dfrac{1}{8}$.

The second equality appeals to the product-to-sum formulas

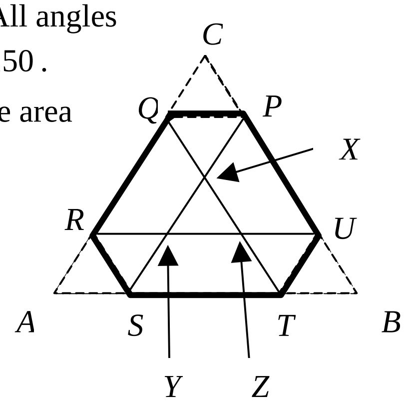
$\sin u \cos v = \frac{1}{2}[\sin(u + v) + \sin(u - v)]$, and $\cos u \sin v = \frac{1}{2}[\sin(u + v) - \sin(u - v)]$.

The fourth equality is based on the Law of Sines, $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$, which, simply put, says that the proportion $\sin A : \sin B : \sin C$ is the same as $a : b : c$.

17. [E] The plane 3 m above the base forms the base of a pyramid whose dimensions are a factor of $\frac{9-3}{9}=\frac{2}{3}$ times those of the original, with a volume $\left(\frac{2}{3}\right)^3=\frac{8}{27}$ of the original. So the answer is $\left(1-\frac{8}{27}\right)\cdot\left[\frac{1}{3}(6\,\text{m})^2(9\,\text{m})\right]=76\,\text{m}^3$.

18. [B] The area of ΔDEF is $\frac{1}{2}(\overline{DE})^2\sin D=\frac{1}{2}(\frac{1}{2}\cdot\overline{AB})^2\sin(2A)=\frac{1}{8}(\overline{AB})^2 2\sin A\cos A$ $=\frac{1}{2}\cos A\left[\frac{1}{2}(\overline{AB})^2\sin A\right]$, which is $\frac{1}{2}\cos A$ times the area of ΔABC. Thus the area of ΔABC is $1/(\frac{1}{2}\cos A)=2\sec A$ times the area of ΔDEF.

19. [A] The accompanying picture illustrates the situation at hand. All angles are either $60°$ or $120°$. $\overline{RU}=\overline{CR}=\overline{CQ}+\overline{QR}=50+100=150$. But $\overline{RY}=\overline{RS}=50$, and likewise $\overline{ZU}=50$, so $\overline{YZ}=50$. The area of the equilateral triangle ΔXYZ is thus $\frac{1}{2}(50)^2\sin(60°)=\frac{1}{2}(50)^2\frac{\sqrt{3}}{2}=625\sqrt{3}\approx 1082.5$.



20. [C] Start with a 3-term geometric series x^2+x+1. Multiplying it by it's alternating counterpart, we get $(x^2+x+1)(x^2-x+1)=(x^2+1)^2-x^2=x^4+x^2+1$, which resembles the original x^2+x+1, with x replaced by its own square. Likewise, by multiplying by its alternating counterpart x^4-x^2+1, we get $(x^4+x^2+1)(x^4-x^2+1)=x^8+x^4+1$. Similarly, $(x^8+x^4+1)(x^8-x^4+1)=x^{16}+x^8+1$. Repeating in this manner, upon eventually multiplying by $x^{2^{k+1}}-x^{2^k}+1$, i.e. $\left(x^{2^k}\right)^2-x^{2^k}+1$, we will get $\left(x^{2^k}\right)^4+\left(x^{2^k}\right)^2+1$. Subtracting 1 to get $\left(x^{2^k}\right)^4+\left(x^{2^k}\right)^2$. With 2 substituting for x, we get $P(k)=\left(2^{2^k}\right)^4+\left(2^{2^k}\right)^2=\left(2^{2^k}\right)^2\left[\left(2^{2^k}\right)^2+1\right]=2^{2^{k+1}}(2^{2^{k+1}}+1)$. So $n=2^{2^{k+1}}$, thus $n+(n+1)=2(2^{2^{k+1}})+1=2^{2^{k+1}+1}+1$. Hence $n+(n+1)\geq 10^{1000}$ means $2^{2^{k+1}+1}+1\geq 10^{1000}$. This is the same as $2^{2^{k+1}+1}\geq 10^{1000}$ since both $2^{2^{k+1}+1}$ and 10^{1000} are even. This says $(2^{k+1}+1)\ln 2\geq 1000\ln 10$, namely $2^{k+1}\geq 1000\frac{\ln 10}{\ln 2}-1$, i.e. $(k+1)\ln 2\geq\ln(1000\frac{\ln 10}{\ln 2}-1)$, i.e. $k\geq\frac{1}{\ln 2}\ln(1000\frac{\ln 10}{\ln 2}-1)-1\approx 10.69737$. This says $k\geq 11$.

1. If $g(x - 1) = x^2 + 1$, find $g(2)$.

A. 1 B. 2 C. 5 D. 9 E. 10

2. Airport runways are labeled by two numbers giving the nonnegative clockwise angles less than 360° of the runway's direction measured from north to the nearest 10°, divided by 10. Thus a runway with heading 223° is labeled 22. What is the other number on this runway?

A. 4 B. 14 C. 16 D. 32 E. 40

3. The equation $a^3 + b^3 + c^3 = 2008$ has a solution in which a, b, and c are distinct even positive integers. Find $a + b + c$.

A. 20 B. 22 C. 24 D. 26 E. 28

4. For how many different integers b is the polynomial $x^2 + bx + 16$ factorable over the integers?

A. 2 B. 3 C. 4 D. 5 E. 6

5. Let $f(x) = x^2 - 2x + 4$. Which of the following is a factor of $f(x) - f(2y)$?

A. $x + 2y$ B. $x + 2y + 2$ C. $x - 2y + 2$ D. $x + 2y - 2$ E. none of these

6. In square $MATH$, M and A lie on a circle of radius 20, and the circle is tangent to side $\overline{TH}$ at the midpoint of $\overline{TH}$. Find the lengths of the sides of the square.

A. 24 B. 26 C. 28 D. 30 E. 32

7. A fair coin is labeled A on one side and M on the other; a fair die has two sides labeled T, two labeled Y, and two labeled C. The coin and die are each tossed three times. Find the probability that the six letters can be arranged to spell AMATYC.

A. $\frac{1}{60}$ B. $\frac{1}{48}$ C. $\frac{1}{36}$ D. $\frac{1}{24}$ E. $\frac{1}{12}$

8. What is the value of $(\log_{624} 625)(\log_{623} 624)(\log_{622} 623)\ldots(\log_6 7)(\log_5 6)$?

A. 2 B. 2.5 C. 4 D. 5 E. 6

9. The letters AMATYC are written in order, one letter to a square of graph paper, to fill 100 squares. If three squares are chosen at random without replacement, find the probability to the nearest 1/10 of a percent of getting three A's.

A. 3.3% B. 3.7% C. 4.0% D. 7.3% E. 11.1%

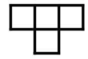
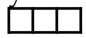
10. A student committee must consist of two seniors and three juniors. Five seniors are able to serve on the committee. What is the least number of junior volunteers needed if the selectors want at least 600 different possible ways to pick the committee?

A. 6 B. 7 C. 8 D. 9 E. 10

11. Ed drives to work at a constant speed S. One day he is halfway to work when he immediately turns around, speeds up by 8 mph, and drives home. As soon as he is home, he turns around and drives to work at 6 mph faster than he drove home. His total driving time is exactly 67% greater than usual. Find S in mph and write the answer in the corresponding blank on the answer sheet.

12. Each bag to be loaded onto a plane weighs either 12, 18, or 22 lb. If the plane is carrying exactly 1000 lb of luggage, what is the largest number of bags it could be carrying?

A. 80 B. 81 C. 82 D. 83 E. 84

13. An 8x8 checkerboard is exactly covered by 16  shaped tiles. What is the least possible number of tiles for which the  is horizontal?

A. 0 B. 2 C. 4 D. 6 E. 8

14. Call a positive integer *biprime* if it is the product of exactly two distinct primes (thus 6 and 15 are biprime, but 9 and 12 are not). If N is the smallest number such that N, N + 1, and N + 2 are all biprime, find the largest prime factor of N(N + 1)(N + 2).

A. 13 B. 17 C. 29 D. 43 E. 47

15. You have 8 identical red counters and n identical green counters. You find that you can line them up in a single row in such a way that the number of counters whose right-hand neighbor is the same color equals the number of counters whose right-hand neighbor is the other color. What is the largest possible value of n?

A. 17 B. 19 C. 21 D. 25 E. 27

16. If b and c are positive integers such that $b/11$, c/b, and $c/15$ all lie in the interval (1.5, 1.8), find $b + c$.

A. 43 B. 44 C. 45 D. 46 E. 47

17. Let r, s, and t be nonnegative integers. For how many such triples (r, s, t) satisfying the system $\begin{cases} rs + t = 24 \\ r + st = 24 \end{cases}$ is it true that $r + s + t = 25$?

A. 23 B. 24 C. 25 D. 26 E. 27

18. In ΔABC, AB = AC = 25 and BC = 14. The perpendicular distances from a point P in the interior of ΔABC to each of the three sides are equal. Find this distance.

A. $\frac{9}{2}$ B. $\frac{19}{4}$ C. 5 D. $\frac{21}{4}$ E. $\frac{11}{2}$

19. The digits 1 to 9 can be separated into 3 disjoint sets of 3 digits each so that the digits in each set can be arranged to form a 3-digit perfect square. Find the last two digits of the sum of these three perfect squares.

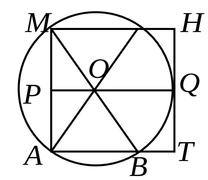
A. 26 B. 29 C. 34 D. 46 E. 74

20. The sequence $\{a_n\}$ is defined by $a_0 = a_1 = a_2 = 1$, and $a_{n-3}a_n - a_{n-2}a_{n-1} = (n-3)!$ for $n \ge 3$. If 5^k is the largest power of 5 that is a factor of $a_{100}a_{101}$, find k.

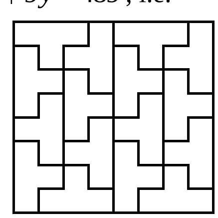
A. 20 B. 22 C. 24 D. 25 E. 26

AMATYC Contest (Spring 2008) SOLUTIONS

1. [E] $g(2) = g(3-1) = 3^2 + 1 = 10$
2. [A] The opposite direction is $223^\circ - 180^\circ = 43^\circ$, which gives the number 4.
3. [B] Let $a = 2x$, $b = 2y$, $c = 2z$. Then $x^3 + y^3 + z^3 = 251$. Since $1^3 = 1$, $2^3 = 8$, $3^3 = 27$, $4^3 = 64$, $5^3 = 125$, $6^3 = 216$, $7^3 = 343$, we see that x, y, z are at most 6. Moreover, since $3 \cdot 64 = 192 < 251$, the largest of x, y, z must be 5 or 6. Say, $x = 6$, then $y^3 + z^3 = 251 - 216 = 35$, which is $27 + 8 = 3^3 + 2^3$. So $x + y + z = 6 + 3 + 2 = 11$, i.e. $a + b + c = 22$. It is not hard to see that this choice of x, y, z (and thus a, b, c) is the only possibility up to reordering.
4. [E] $16 = (1)(16) = (2)(8) = (4)(4)$, so b can be ± 17, ± 10, ± 8.
5. [D] $f(x) - f(2y) = \cdots = x^2 - 4y^2 - 2(x - 2y) = (x-2y)(x+2y) - 2(x-2y) = (x-2y)(x+2y-2)$
6. [E] Let O be the center of the circle. Let B be the point between A and T that lies on the circle. Let Q be where the circle is tangent to $\overline{TH}$, and P where the ray $\overrightarrow{QO}$ intersects $\overline{MA}$. Then $\angle MOP = \angle AOP = \angle OAB = \angle OBA = \angle BOQ$. Thus M, O, B lie on the same line, and so $MB = 40$. The Tangent-Secant Theorem of a circle says $(AT)(BT) = (QT)^2$. But $AT = 2(QT)$, so $BT = (QT)/2 = (AT)/4$. It follows that $AB = \frac{3}{4}(AT) = \frac{3}{4}(AM)$, so the right triangle MBA is a 3:4:5 right triangle, and so $AM = \frac{4}{5}(MB) = \frac{4}{5}(40) = 32$. (It's also possible to take an algebraic approach.)



7. [E] Among the 2^3 equally likely outcomes of the three tosses of the coin, 3 have exactly two A's turned up. Among the 3^3 equally likely outcomes of the three tosses of the die, $3! = 6$ have exactly one T, one Y, and one C. So the answer is $\frac{(3)(3!)}{(2^3)(3^3)} = \frac{1}{12}$
8. [C] $\frac{\ln 625}{\ln 624} \cdot \frac{\ln 624}{\ln 623} \cdots \frac{\ln 7}{\ln 6} \cdot \frac{\ln 6}{\ln 5} = \frac{\ln 625}{\ln 5} = \log_5 625 = 4$.
9. [B] The complete sequence AMATYC occurs 16 times, taking up $16 \times 6 = 96$ squares. The remaining four squares are AMAT. Thus there are $2 \times 16 + 2 = 34$ A's out of 100 squares. The answer is thus $\frac{34}{100} \cdot \frac{33}{99} \cdot \frac{32}{98} \approx 0.0370068 \approx 3.7\%$.
10. [D] Let there be n juniors. Want $C_2^5 C_3^n \geq 600$, i.e. $\frac{5 \times 4}{1 \times 2} \cdot \frac{n(n-1)(n-2)}{1 \times 2 \times 3} \geq 600$, namely $n(n-1)(n-2) \geq 360$. Starting from $n = 3$, $n(n-1)(n-2)$ goes up as n increases. Direct computation quickly shows that it is $n = 9$ when it first reaches at least 360.
11. [42] $\frac{1/2}{S} + \frac{1/2}{S+8} + \frac{1}{S+14} = \frac{167}{100}\frac{1}{S}$, which can be brought to $11S^2 - 358S - 4368 = 0$. Then use the quadratic formula.
12. [C] Let there be x, y, z bags that weigh 12, 18, and 22 lb, respectively. So $12x + 18y + 22z = 1000$, i.e. $6x + 9y + 11z = 500$. But $1(6) + 3(9) = 3(11)$. Thus, replacing 3 22-lb bags by 1 12-lb bag and 3 18-lb bags would increase the number of bags without changing the total weight. So when the number of bags is maximized, z has to be 0 or 1 or 2. Likewise, since $3(6) = 2(9)$, y has to be 0 or 1. Since $6x + 9y$ is divisible by 3, so is $500 - 11z$, thus $z = 1$. Therefore $6x + 9y = 489$, i.e. $2x + 3y = 163$, and so y has to be odd, thus $y = 1$. So $(x, y, z) = (80, 1, 1)$.
13. [C] The number of horizontal tiles can be 4, as illustrated. It can be reasoned in a straightforward manner that neither 0 nor 2 horizontal blocks are possible.



14. [B] Observe that a biprime cannot be a multiple of 4. Hence N has to be congruent to 1 modulo 4 (i.e. of the form $4k+1$), for otherwise at least one of the three numbers would be a multiple of 4. It follows that $N+1$ is even. But $N+1$ is coprime, so $N+1 = 2p$, with $p \neq 2$ a prime, and thus $N+1 =$ 6, 10, 14, 22, 26, 34, ….

We quickly rule out $N+1=6$, through 26, as either N or $N+2$ would fail to be a coprime. Thus N, $N+1$, $N+2$ are 33, 34, 35, with $N(N+1)(N+2)=3\cdot 11\cdot 2\cdot 17\cdot 7\cdot 5$. The answer follows. Alternatively, list the coprimes one by one until three contiguous numbers first appear.

15. [D] Let there be x counters whose right-hand neighbor is the same color. So there are also x counters whose right-hand neighbor is the same color. Then there will be a total of $2x+1$ blocks. Thus $2x+1=8+n$. To maximize n means to maximize x. There will be $x+1$ groups of blocks, with blocks within each group being of the same color, and the color alternates from one group to the next. Since there are 8 red counters, x can be <u>at most</u> 16, with the group pattern being GRGRGRGRGRGRGRGRG. For this pattern, the R groups all would contain only one block. To have x counters whose right-hand neighbor is the same color, we have to start with each G group being of one single (green) block, then add x new green blocks (which can be distributed in any manner to the G groups.) This realizes the case $x=16$, $n=2x+1-8=25$.

16. [A] $1.5<b/11<1.8$ gives $16.5<b<19.8$, i.e. $17\le b\le 19$. Likewise, $1.5<c/15<1.8$ gives $23\le c\le 26$. For b and c in these ranges, the highest ratio c/b is $26/17=1.5294...$ and other combinations have the ratio c/b below 1.5. The answer is thus $26+17=43$.

17. [D] Take the difference of the two equations in the system to get $r(s-1)+t(1-s)=0$, i.e. $(s-1)(r-t)=0$. <u>Case 1</u>: $s=1$. Thus the system says $r+t=24$ and $r+s+t=25$ also becomes $r+t=24$. There are 25 possible triples (r,s,t) of this kind. <u>Case 2</u>: $s\ne 1$, thus $r=t$. The system says $t(s+1)=24$, while $r+s+t=25$ says $2t+s=25$. So we have $\begin{array}{rcl} t(s+1) & = & 24 \\ 2t+s & = & 25 \end{array}$, which means $\begin{array}{rcl} (2t)(s+1) & = & 48 \\ (2t)+(s+1) & = & 26 \end{array}$, therefore $2t$ and $s+1$ form the two roots of the quadratic equation $x^2-26x+48=0$, i.e. $(x-2)(x-24)=0$. Since $s\ne 1$, it follows that $s+1=24$, $2t=2$, so $(r,s,t)=(1,23,1)$. Cases 1 and 2 combined give 26 triples.

18. [D] Let D be the midpoint of $\overline{BC}$. Apply the Pythagorean Theorem to ΔABD to get $AD=24$, and so the areas of ΔABC is $\frac{1}{2}(14)(24)=168$. On the other hand, the area of ΔABC is the sum of the areas of ΔPAB, ΔPBC, ΔPCA, and so is $\frac{1}{2}(AB+BC+CA)r=32r$, where r is the distance we are seeking. So $168=32r$, $r=21/4$.

19. [E] Approach the problem by brute force. First, list all perfect squares formed by three distinct non-zero digits by squaring 10, 11, 12, etc and ruling out those with repeated digits. The resulting list is 169, 196, 256, 289, 324, 361, 529, 576, 625, 729, 784, 841, 961. Among them only 324 and 361 contain the digit 3, thus one of them has to be included. If it's 324, then the other two perfect squares must be from among 169, 196, 576, 961, in order to avoid reusing 3, 2, or 4. For the digit 5 to appear, we are forced to include 576, with the remaining digits 1, 8, 9 unable to form a perfect square, thus 324 doesn't work. Try 361, with the other two perfect squares thus coming from among 289, 529, 729, 784. To accommodate the digit 5, we include 529, with the remaining perfect square thus being 784. We have $361+529+784=1674$.

20. [C] Divide $a_{n-3}a_n-a_{n-2}a_{n-1}=(n-3)!$ by $a_{n-3}a_{n-2}$ on both sides to get $\frac{a_n}{a_{n-2}}-\frac{a_{n-1}}{a_{n-3}}=\frac{(n-3)!}{a_{n-3}a_{n-2}}$. Start with $\frac{a_2}{a_0}=1$, from $\frac{a_3}{a_1}-\frac{a_2}{a_0}=\frac{0!}{a_0a_1}$, get $\frac{a_3}{a_1}-1=\frac{0!}{(1)(1)}$, and so $\frac{a_3}{a_1}=2$. Likewise, $\frac{a_4}{a_2}-\frac{a_3}{a_1}=\frac{1!}{a_1a_2}$ gives $\frac{a_4}{a_2}=3$. To continue, for $n\ge 5$, rewrite $\frac{a_n}{a_{n-2}}-\frac{a_{n-1}}{a_{n-3}}=\frac{(n-3)!}{a_{n-3}a_{n-2}}$ as

$\frac{a_n}{a_{n-2}}-\frac{a_{n-1}}{a_{n-3}}=\frac{(n-3)!}{a_0a_1\frac{a_2}{a_0}\cdot\ldots\cdot\frac{a_{n-3}}{a_{n-5}}\cdot\frac{a_{n-2}}{a_{n-4}}}$ and so $\frac{a_n}{a_{n-2}}-\frac{a_{n-1}}{a_{n-3}}=\frac{(n-3)!}{\frac{a_2}{a_0}\cdot\ldots\cdot\frac{a_{n-3}}{a_{n-5}}\cdot\frac{a_{n-2}}{a_{n-4}}}$. So if $\frac{a_k}{a_{k-2}}=k-1$ for $2\le k\le n-1$, then RHS is 1, and $\frac{a_n}{a_{n-2}}=\frac{a_{n-1}}{a_{n-3}}+1=n-2+1=n-1$. So in general $\frac{a_n}{a_{n-2}}=n-1$. It follows that $a_{100}=(1)(3)\cdots(99)$ and $a_{101}=(2)(4)\cdots(100)$. So $a_{100}a_{101}=(1)(2)\cdots(100)$. Twenty of the one hundred factors on the RHS are divisible by 5. Four among the twenty each contains a prime factor 5 twice, the rest being divisible by 5 once only. So the answer is $20+4=24$.

1. If the coordinates of one endpoint of a line segment are (3, -3) and the coordinates of the midpoint are (7, 5), what are the coordinates of the other endpoint?

A. (11, 13) B. (13, 11) C. (17, 7) D. (7, 17) E. (5, 1)

2. Let the operation Δ be defined for positive integers a and b by aΔb = ab + b. If xΔ(x - 1) = 323, find xΔ(x + 1).

A. 324 B. 325 C. 342 D. 360 E. 361

3. The perimeter of a rectangle is 36 ft and a diagonal is $\sqrt{170}$ ft. Its area in ft^2 is

A. 70 B. 72 C. 75 D. 77 E. 80

4. Which of the following functions satisfies the equation f(x + f(x)) = f(f(x)) + f(x) for all real values of x and y?

A. f(x) = x B. f(x) = 2x C. f(x) = ln (x) D. A and B E. all of them

5. For what values of k will the equation $x\sqrt{14}+7=kx^2$ have exactly 2 real solutions?

A. $k > 2$ B. $k > -1/2$ C. $k > -2$ D. $k < -2$ E. $k < -1/2$

6. If x and n are positive integers with $x > n$ and $x^n - x^{n-1} - x^{n-2} = 2009$, find $x + n$.

A. 10 B. 11 C. 12 D. 13 E. 14

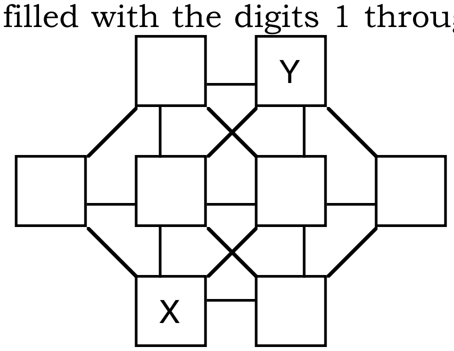
7. In a tournament, 3/7 of the women are matched against half of the men. What fraction of all the players is matched against someone of the other gender?

A. 2/5 B. 3/7 C. 4/9 D. 6/13 E. 13/28

8. Four points *A*, *B*, *C*, and *D* on a given circle are chosen. If the diagonals of quadrilateral *ABCD* intersect at the center of the circle, then *ABCD* must be a

A. trapezoid B. square C. rectangle D. kite E. none of these

9. In the diagram shown, the boxes are to be filled with the digits 1 through 8 (each used exactly once). If no two boxes connected directly by a line segment can contain consecutive digits, find X + Y.



A. 7 B. 8 C. 9
D. 10 E. 11

10. A cone has a circular base with a radius of 4 cm. A slice is made parallel to the base of the cone so that the new cone formed has half the volume of the original cone. What is the radius in centimeters of the base of the new cone?

A. $2\sqrt[3]{4}$ B. $2\sqrt[3]{2}$ C. $2\sqrt{2}$ D. 2 E. 1

11. At one point as Elena climbs a ladder, she finds that the number of rungs above her is twice the number below her (not counting the rung she is on). After climbing 5 more rungs, she finds that the number of rungs above and below her are equal. How many more rungs must she climb to have the number below her be four times the number above her?

A. 5 B. 6 C. 7 D. 8 E. 9

12. If sin θ - cos θ = 0.2 and sin 2θ = 0.96, find $\sin^3\theta - \cos^3\theta$.

A. 0.25 B. 0.276 C. 0.28 D. 0.296 E. 0.30

13. How many asymptotes does the function $g(x) = \dfrac{x}{10\sqrt{100x^2 - 1}}$ have?

A. 0 B. 1 C. 2 D. 3 E. 4

14. For how many solutions of the equation $x^2 + 4x + 6 = y^2$ are both x and y integers?

A. 0 B. 1 C. 2 D. 3 E. an infinite number

15. The sum of the squares of the four integers r, s, t, and u is 685, and the product of r and s is the opposite of the product of t and u. Find $|r| + |s| + |t| + |u|$.

A. 39 B. 41 C. 43 D. 45 E. 47

16. You pass through five traffic signals on your way to work. Each is either red, yellow, or green. A red is always immediately followed by a yellow; a green is never followed immediately by a green. How many different sequences of colors are possible for the five signals?

A. 42 B. 48 C. 54 D. 60 E. 66

17. How many different ordered pairs of integers with $y \neq 0$ are solutions for the system of equations $6x^2y + y^3 + 10xy = 0$ and $2x^2y + 2xy = 0$?

A. 1 B. 2 C. 3 D. 4 E. 5

18. The graph of the equation $x + y = x^3 + y^3$ is the union of a

A. line and an ellipse B. line and a parabola C. parabola and an ellipse
D. pair of lines E. line and a hyperbola

19. A four-digit number each of whose digits is 1, 5, or 9 is divisible by 37. If the digits add up to 16, find the sum of the last two digits.

A. 2 B. 6 C. 10 D. 12 E. 14

20. In ΔABC, $AB = 5$, $BC = 8$, and $\angle B = 90°$. Choose D on $\overline{AB}$ and E on $\overline{BC}$ such that $BD = 3$ and $BE = 5$. Find the area common to the interiors of ΔABC and the rectangle determined by $\overline{BD}$ and $\overline{BE}$.

A. 1111/80 B. 1113/80 C. 1117/80 D. 1119/80 E. 1121/80

Laney College

AMATYC Contest (Spring 2009) SOLUTIONS

1. [A] $(2\cdot 7-3,\ \ 2\cdot 5-(-3))=(11,13)$.
2. [E] $a\Delta b=ab+b=(a+1)b$, so $323=x\Delta(x-1)=(x+1)(x-1)=x^2-1$, thus $x^2=324=4\cdot 81$, so the positive integer x is $2\cdot 9=18$. Therefore $x\Delta(x+1)=(x+1)(x+1)=19\cdot 19=361$.
3. [D] Length ℓ in, width w in. So $\ell+w=36/2=18$ and $\ell^2+w^2=170$. Thus $\ell w=\frac{1}{2}((\ell+w)^2-(\ell^2+w^2))=\frac{1}{2}(18^2-170)=\frac{1}{2}(324-170)=77$.
4. [D] It's straightforward to check that A and B work. For C, it amounts to checking $\ln(x+\ln x)$ is different from $\ln(\ln x)+\ln x$, which can be seen by, say, $x=e$.
5. [B] Rewrite the equation as $kx^2-x\sqrt{14}-7=0$. This quadratic equation has two real solutions precisely when $(-\sqrt{14})^2-4(k)(-7)>0$, i.e. $k>-1/2$.
6. [B] Rewrite as $x^{n-2}(x^2-x-1)=2009$. But $2009=7^2\cdot 41$. Clearly $x=7$ and $n=4$ would work. (It is not hard to see that this is the only possibility.)
7. [D] The four numbers (# women matching against men), (# remaining women), (# men matching against women), (# remaining men) is of the proportion $3:4:3:3$. So the answer is $(3+3)/(3+4+3+3)=6/13$.
8. [C] (Draw a picture, and it would be obvious.)
9. [C] The box right below Y is connected to all other boxes except that at the far left; so it has to hold 1 or 8. Suppose it's 8, then the box at the far left is 7. The same argument applied to the box right above X shows it holds 1, with the box at the far right holding 2. Of the four remaining boxes forming a rectangle, 3 should be on the left column (to stay away from 2) while 6 on the right, leaving 4 and 5 on separate columns too. So 4 and 5 have to be on one diagonal since they shouldn't be joined. Thus 3 and 6 occupy the other diagonal. It follows that $X+Y=9$.
10. [A] $(4cm)\cdot\sqrt[3]{1/2}$, i.e. $2\sqrt[3]{4}\,cm$.
11. [E] Let there be n rungs both above and below at the second moment stated. Thus $n+5=2(n-5)$, so $n=15$. At the third moment, of the 30 rungs other than the rung Elena is on, there should be therefore 6 above and 24 below. The answer is therefore $15-6=9$.
12. [D] $\sin^3\theta-\cos^3\theta=(\sin\theta-\cos\theta)(\sin^2\theta+\sin\theta\cos\theta+\cos^2\theta)$

    $=(\sin\theta-\cos\theta)(1+\sin\theta\cos\theta)=(0.2)(1+\frac{1}{2}2\sin\theta\cos\theta)=(0.2)(1+\frac{1}{2}\sin 2\theta)$

    $=(0.2)(1+0.96/2)=0.296$
13. [E] Two vertical asymptotes and two horizontal asymptotes: $x=1/10$, $x=-1/10$, $y=1/100$, $y=-1/100$.
14. [A] Rewrite the equation as $2(2x+3)=(y+x)(y-x)$. Note that $2x+3$ is a nonzero odd number if x is an integer. Thus $(y+x)(y-x)$ has exactly one factor of 2. This is impossible, as $y+x$ and $y-x$ are either both odd or both even.
15. [Erroneous] If r, s, t, u are all nonzero, $rs=-tu$ would imply that the ratio $|r|:|t|$ equals the ratio $|u|:|s|$. So $|r|=ax$, $|t|=ay$, $|u|=bx$, $|s|=by$, where a, b, x, y are positive integers, with x and y relatively prime. Then

$r^2 + s^2 + t^2 + u^2 = 685$ reads $(a^2 + b^2)(x^2 + y^2) = (5)(137)$. (Note that 137 is a prime.) So $a^2 + b^2 = 5$ and $x^2 + y^2 = 137$, or the other way around. For $a^2 + b^2 = 5$ and $x^2 + y^2 = 137$, it follows that $\{a. b\} = \{1, 2\}$ while $\{x. y\} = \{4, 11\}$. (Remark: There is a theorem stating that any PRIME number of the form $4k+1$ can be expressed as the sum of two perfect squares. This ensures that 137 can be expressed as the sum of two perfect squares.) Hence $|r|$, $|t|$, $|u|$, $|s|$ are 4, 11, 8, 22 (not necessarily in that order), so $|r| + |s| + |t| + |u| = 45$. This is probably the answer originally intended. The trouble lies in the possibility for some of the four numbers to be zero. This allows for cases where two of the numbers are zero, while the absolute values of the remaining two are $\{3, 26\}$ or $\{18, 19\}$.

16. [D] R either bundled with Y (in that order), or else has to appear at the far right.
    Case 1: (RY)(RY)R [1]
    Case 2: *(RY)(RY) [2], (RY)*(RY) [2], (RY)(RY)* [2]
    Case 3: **(RY)R [3], *(RY)*R [4], (RY)**R [3]
    Case 4: ***(RY) [$1+3+1=5$], (RY)*** [5], **(RY)* [$3\times 2 = 6$], *(RY)** [6]
    Case 5: ****R [$1+4+(2+1)=8$]
    Case 6: ***** [$1+5+(3+2+1)+1=13$]
    $1+2+2+2+3+4+3+5+5+6+6+8+13=60$
17. [B] Because $y \neq 0$, the first equation implies $6x^2 + y^2 + 10x = 0$ and the second equation implies $x^2 + x = 0$. But $x^2 + x = 0$ means $x(x+1) = 0$, so $x = 0, -1$. But $x = 0$ would make $6x^2 + y^2 + 10x = 0$ into $y = 0$, which is not the case. So $x = -1$, in which case $6x^2 + y^2 + 10x = 0$ becomes $y^2 - 4 = 0$, so $y = \pm 2$.
18. [A] $x + y = x^3 + y^3$ is equivalent to $x + y = (x+y)(x^2 - xy + y^2)$, i.e. $(x+y)(x^2 - xy + y^2 - 1) = 0$. So the graph is the union of the line $x + y = 0$ and the conic section $x^2 - xy + y^2 - 1 = 0$, which can be rotated into one of the form $\alpha x^2 + \beta y^2 - 1 = 0$, with α and β the eigenvalues of the symmetric matrix $\begin{bmatrix} 1 & -1/2 \\ -1/2 & 1 \end{bmatrix}$, i.e. solutions of $(1-t)(1-t) - (-1/2)(-1/2) = 0$, i.e. $\frac{1}{2}$ and $\frac{3}{2}$. It follows that $x^2 - xy + y^2 - 1 = 0$ is a parabola.
19. [C] The four digits must be 9511 or 5551, or their permutations. There are therefore 16 possibilities. Try them one by one, you have 1591. In the case of hand computation, rather than using long division, it would be faster to use the following divisibility test for 37: Given a number Xy, where y is the ones digit, while X stands for the number after y is removed, the number Xy is divisible by 37 if and only if X-11y (i.e. 11 times y, subtracted from X) is divisible by 37.
20. [D] The area region common to the right triangle and the rectangle equals the area of the right triangle with the two smaller triangles (shaded) removed. The two smaller right triangles are both similar to the original right triangle. The answer is $\frac{1}{2}(8)(5)\left[1 - \left(\frac{2}{5}\right)^2 - \left(\frac{3}{8}\right)^2\right] = 20\left[1 - \frac{4}{25} - \frac{9}{64}\right] = 20 \cdot \frac{1600-256-225}{1600} = \frac{1119}{80}$.